Calculate the limit
https://www.linkedin.com/feed/update/urn:li:activity:6452843082613428224
Calculate the limit
$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n}$
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Solution 1.
Since $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$ (follows from inequality $\left(\frac{n+1}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^n (n+1) \iff$
$\frac{n+1}{en} < \frac{\sqrt[n]{n!}}{n} < \frac{n+1}{en} \cdot \sqrt[n]{n+1}$ and Squeeze Principle) and
$(2n-1)!! = \frac{(2n)!}{2^n n!}$ we obtain $\frac{\sqrt[n]{(2n-1)!!}}{n} = \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \sqrt[n]{\frac{(2n)!}{2^n \cdot n^n \cdot n!}} =$
$\sqrt[n]{\frac{(2n)! \cdot 2^n \cdot n^n}{(2n)^{2n} \cdot n!}} = 2\sqrt[n]{\frac{n^n}{n!}} \cdot \left(\sqrt[2n]{\frac{(2n)!}{(2n)^{2n}}}\right)^2 = 2 \cdot \frac{n}{\sqrt[n]{n!}} \cdot \left(\frac{\sqrt[2n]{(2n)!}}{2n}\right)^2.$
Therefore, $\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = 2\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} \cdot \lim_{n\to\infty}\left(\frac{\sqrt[2n]{(2n)!}}{2n}\right)^2 = 2 \cdot e \cdot \frac{1}{e^2} = \frac{2}{e}.$

Solution 2.
Let $a_n := \frac{(2n-1)!!}{n^n}$. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+1)n^n}{(n+1)^{n+1}} =$
$\lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e}$ then by Multiplicative Stolz-Cezaro theorem*
$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{a_n} = \frac{2}{e}.$
* If the sequence $(b_n)_{n\geq 1}$ has a limit, then $\lim_{n\to\infty} \sqrt[n]{b_1 b_2 \ldots b_n} = \lim_{n\to\infty} b_n$.
Then for the sequence $(a_n)_{n\geq 1}$, denoting $b_n := \frac{a_n}{a_{n-1}}, n \in \mathbb{N}$, where $a_0 := 1$,
we obtain $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ if sequence $\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)_{n\geq 1}$ has a limit,.
This theorem can be also named Geometric Mean Limit Theorem (GML theorem)
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